ISRAEL JOURNAL OF MATHEMATICS, Vol. 28, No. 4, 1977

ON BANACH SPACES WHICH CONTAIN $l^1(\tau)$ AND TYPES OF MEASURES ON COMPACT SPACES

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ABSTRACT

Two closely related results are presented, one of them concerned with the connection between topological and measure-theoretic properties of compact spaces, the other being a non-separable analogue of a result of Pełczyński's about Banach spaces containing L^1. Let τ be a regular cardinal satisfying the hypothesis that $\kappa^\omega < \tau$ whenever $\kappa < \tau$. The following are proved: 1) A compact space T carries a Radon measure which is homogeneous of type τ, if and only if there exists a continuous surjection of T onto $[0,1]^\tau$. 2) A Banach space X has a subspace isomorphic to $l^1(\tau)$ if and only if X^* has a subspace isomorphic to $L^1(\{0,1\}^\tau)$. An example is given to show that a more recent result of Rosenthal's about Banach spaces containing l^1 does not have an obvious transfinite analogue. A second example (answering a question of Rosenthal's) shows that there is a Banach space X which contains no copy of $l^1(\omega_1)$, while the unit ball of X^* is not weakly* sequentially compact.

1. Preliminaries

Cardinal numbers $\kappa, \tau \cdots$ will be identified with the corresponding initial ordinals (so that $\kappa = \{\alpha : \alpha$ is an ordinal and $\alpha < \kappa\}$). The *cofinality* $\mathrm{cf}(\kappa)$ of κ is by definition the smallest cardinal λ for which there exists a family $(\kappa_\beta)_{\beta<\lambda}$ with $\kappa_\beta < \kappa$, $\sup_\beta \kappa_\beta = \kappa$. The cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$. When the notation κ^λ is used, it will be cardinal (not ordinal) exponentiation that is intended. The first infinite cardinal will of course be denoted by ω, and the cardinal of a set A by $|A|$.

The Banach spaces considered will all be over the reals as scalar field. Thus $C(T)$ will denote the space of all continuous real-valued functions on the compact (Hausdorff) space T, and the dual $C(T)^*$ will as usual be identified with the space $M(T)$ of all Radon measures on T. If $\varphi : S \to T$ is continuous, φ^0 will be the induced map $C(T) \to C(S)$, $\varphi^0(g) = g \circ \varphi$, and I shall write $\tilde{\varphi}$ for the transpose $(\varphi^0)^*$. For a measure $\mu \in M(S)$, $\tilde{\varphi}\mu$ is the usual image measure defined by

$$(\tilde{\varphi}\mu)(B) = \mu(\varphi^{-1}B).$$

Received June 14, 1976 and in revised form October 26, 1976

When μ is a measure, it will be convenient to distinguish the space $\mathscr{L}^1(\mu)$ (consisting of all μ-integrable functions) from $L^1(\mu)$ (the quotient of $\mathscr{L}^1(\mu)$ by the null-functions). For $f \in \mathscr{L}^1(\mu)$, I denote by $f^\bullet$ the corresponding element of $L^1(\mu)$.

If A is any set, $l^1(A)$ is the Banach space of all real valued functions x on A for which $\|x\|_1 = \Sigma_{a\in A} |x(a)|$ is finite. I shall write $\mathbf{D}$ for the two-point set $\{0,1\}$ and λ_A for the usual product measure on the product space $\mathbf{D}^A$. The index set A will very often be a cardinal. According to the Maharam decomposition theorem, any finite measure μ can be expressed as a sum

$$\mu = \sum_{j=1}^{\infty} \nu_j$$

where the ν_j are disjoint, and each ν_j is homogeneous. For a full exposition of this theory, see, for instance, §26.4 of [7]; suffice it to note here that when κ is an infinite cardinal a measure ν is *homogeneous* (of type κ) if and only if $L^1(\nu)$ is isometric to $L^1(\lambda_\kappa)$.

If T is a compact space, then by 19.7.6 of [9] T carries an atomless measure if and only if T has a nonempty perfect subset, or, equivalently, if and only if there exists a continuous surjection $T \to [0,1]$. One of the aims of this paper is to obtain more information about the types of the measures carried by T in terms of the existence of continuous surjections from T onto products $[0,1]^A$. Since every atomless measure on a compact metric space is of type ω, it is with non-metrizable spaces that we shall be concerned.

The other aim is to give a non-separable analogue of the theorem of Pełczyński that a dual Banach space X^* has a subspace isomorphic (that is, linearly homeomorphic) to $L^1[0,1]$ if and only if X has a subspace isomorphic to l^1. The obvious nonseparable spaces to look at in this connection are $L^1(\lambda_\tau)$ and $l^1(\tau)$, where τ is an uncountable cardinal.

The notion of an independent family of pairs of sets, introduced by Rosenthal in his work on Banach spaces which contain l^1 [8], turns out to be critically important in both the contexts considered here. A family $(A^0_\alpha, A^1_\alpha)_{\alpha\in\sigma}$ of pairs of sets will be called *independent* if

(i) for each α $A^0_\alpha \cap A^1_\alpha = \varnothing$,

(ii) for every finite subset M of σ and every map $\varepsilon : M \to \{0,1\}$

$$\bigcap_{\alpha\in M} A^{\varepsilon(\alpha)}_\alpha \neq \varnothing.$$

The importance of such families here depends on the following two lemmas.

1.1 Lemma. *For a compact space T, and an infinite cardinal τ, the following are equivalent*:

(a) *there is an independent family $(A^0_\alpha, A^1_\alpha)_{\alpha\in\tau}$ consisting of closed subsets of T*;

(b) *there is a closed subset S of T and a continuous surjection $\varphi : S \to \{0,1\}^\tau$*;

(c) *there is a continuous surjection $\psi : T \to [0,1]^\tau$*.

1.2 Lemma. *Let S be a set and $(x_\alpha)_{\alpha\in\sigma}$ be a uniformly bounded family of real-valued functions on S. If, for some real r, δ with $\delta > 0$, the sets*

$$A^0_\alpha = \{s \in S : x_\alpha(s) \geqq r + \delta\},$$

$$A^1_\alpha = \{s \in S : x_\alpha(s) \leqq r\}$$

form an independent family, then $(x_\alpha)_{\alpha\in\sigma}$ is equivalent to the usual (transfinite) basis of $l^1(\sigma)$.

Proof. This is proposition 4 of [7].

The technique for finding independent families will depend on a combinatorial principle of Erdös and Rado (and it is here that the restriction on the cardinal τ comes in). Recall that a family of sets $(E_\alpha)_{\alpha\in\sigma}$ is said to be *quasidisjoint* (or to be a Δ-system) if $E_\alpha \cap E_\beta = \bigcap_{\gamma\in\sigma} E_\gamma$ whenever $\alpha, \beta \in \sigma$ and $\alpha \neq \beta$.

1.3 Lemma. *Let τ be a regular cardinal with the property that $\kappa^\omega < \tau$ whenever $\kappa < \tau$, and let $(E_\alpha)_{\alpha\in\tau}$ be a family of countable sets. Then there is a subset σ of τ with $|\sigma| = \tau$, such that $(E_\alpha)_{\alpha\in\sigma}$ is quasidisjoint.*

Proof. See theorem I of [1] or the appendix of [5].

The smallest cardinal τ for which the hypotheses of 1.3 are satisfied is the successor of the continuum $\tau = (2^\omega)^+$. Subject to the generalized continuum hypothesis, they are satisfied for a successor cardinal $\tau = \kappa^+$ if and only if $\mathrm{cf}(\kappa) > \omega$. The idea of using a combinatorial result like 1.3 in the present context was suggested by Hagler's work on dyadic spaces.

I should like to thank Professor Hagler for making available preprints of his papers [2] and [3]. My thanks are also due to the referee for some valuable suggestions, especially for the easy proof of 2.1.

2. The main results

First let us notice that in each of the theorems to be proved the implication in one direction is valid without special assumptions on cardinality. The arguments used in the next two propositions should be familiar.

2.1 Proposition. *Let R, T be compact spaces and $\varphi : T \to R$ be a continuous surjection. For each $\lambda \in M_+(R)$ there is a $\nu \in M_+(T)$ with $\tilde{\varphi}\nu = \lambda$ and such that $L^1(\nu)$ is isometric to $L^1(\lambda)$.*

Proof. Let A denote $\{\mu \in M_+(T) : \tilde{\varphi}\mu = \lambda\}$. Then A is a nonempty, weakly* compact, convex subset of $M(T)$. The map $\varphi^0 : L^1(\nu) \to L^1(\lambda)$ is an isometric embedding for any $\nu \in A$. We shall show that it is surjective provided ν is an extreme point of A. Suppose then that $\varphi^0 L^1(\lambda) \neq L^1(\nu)$; there exists a nonzero element g of ball $L^1(\nu)$ with $\int (\varphi^0 f) . g \, d\nu = 0$ for all $f \in L^1(\lambda)$. It follows from this equality that $\tilde{\varphi}((1+g).\nu) = \tilde{\varphi}((1-g).\nu) = \lambda$. Also, since $|g| \leqq 1$, both of $(1 \pm g).\nu$ are nonnegative measures, and are hence in A. The expression $\nu = \frac{1}{2}[(1+g).\nu + (1-g).\nu]$ tells us that ν is not extreme in A.

2.2 Proposition. *Let X be a Banach space which has a subspace isomorphic to $l^1(\tau)$. Then X^* has a subspace isomorphic to $L^1(\lambda_\tau)$.*

Proof. Since the density character of $L^1(\lambda_\tau)$ is τ, there is an embedding $E : L^1(\lambda_\tau) \to l^\infty(\tau) = l^1(\tau)^*$. If $J : l^1(\tau) \to X$ is an embedding, lemma 2 of [4] allows us to "lift" E to an embedding $F : L^1(\lambda_\tau) \to X^*$ with $J^*F = E$.

We come now to the main technical result. When B is a subset of A let us agree to write π_B for the projection map $\mathbf{D}^A \to \mathbf{D}^B$.

2.3 Proposition. *Let τ be a regular cardinal with the property that $\kappa^\omega < \tau$ whenever $\kappa < \tau$, let A be a set, and let $(f_\alpha^\cdot)_{\alpha \in \tau}$ be a family of elements of $L^\infty(\lambda_A)$ with*

$$\| f_\alpha^\cdot \|_\infty \leqq 1, \ \| f_\alpha^\cdot - f_\beta^\cdot \|_1 \geqq \varepsilon > 0 \qquad (\alpha \neq \beta).$$

Then there exist a subset σ of τ with $|\sigma| = \tau$, and real numbers r, δ with $\delta > 0$, such that the sets

$$A_\alpha^0 = \{z : f_\alpha(z) \geqq r + \delta\},$$

$$A_\alpha^1 = \{z : f_\alpha(z) \leqq r\}$$

form an independent family, in the strong sense that

$$\lambda_\tau \left(\bigcap_{\alpha \in M} A_\alpha^{\varepsilon(\alpha)} \right) \neq 0$$

whenever M is a finite subset of σ, and $\varepsilon \in \mathbf{D}^M$.

Proof. There exist countable subsets $E(\alpha)$ $(\alpha \in \tau)$ of A such that $f_\alpha^\cdot = (g_\alpha \circ \pi_{E(\alpha)})^\cdot$, for suitable $g_\alpha \in \mathscr{L}^1(\lambda_{E(\alpha)})$. Applying the result of Erdös and Rado, quoted here as 1.3, we get a quasidisjoint subfamily $(E(\alpha))_{\alpha\in\sigma_1}$ with $|\sigma_1| = \tau$. Let the common intersection of this subfamily be E, and denote by $\mathscr{E}$ the conditional expectation projection from $L^1(\lambda_A)$ onto $L^1(\lambda_E)$.

Since $\mathrm{cf}(\tau) = \tau > 2^\omega = |L^1(\lambda_E)|$, there is a subset σ_2 of σ_1, with $|\sigma_2| = \tau$, such that $\mathscr{E}f_\alpha^{\cdot}$ is the same element $h^{\cdot}$ of $L^1(\lambda_E)$ for all $\alpha \in \sigma_2$. We can also assume that

$$\| f_\alpha^{\cdot} - (h \circ \pi_E)^{\cdot} \|_1 \geqq \varepsilon/2 \quad \text{for all} \quad \alpha \in \sigma_2.$$

It follows that for each $\alpha \in \sigma_2$ there is a non-null compact subset K_α of $\mathbf{D}^E$ such that

$$\int |g_\alpha(w,z) - h(z)| \, d\lambda_{E(\alpha)\setminus E}(w) \geqq \varepsilon/2$$

for all $z \in K_\alpha$.

Since $g_\alpha(.,z) \in \mathscr{L}^1(\lambda_{E(\alpha)\setminus E})$ and

$$\int g_\alpha(w,z) \, d\lambda_{E(\alpha)\setminus E}(w) = h(z)$$

for almost all $z \in D^E$, we can assume that K_α is chosen so that this equality holds for all $z \in K_\alpha$.

It follows that the following subsets of $\mathbf{D}^{E(\alpha)\setminus E}$ are non-null for all $z \in K_\alpha$:

$$S_\alpha^0(z) = \{w : g_\alpha(w,z) \geqq h(z) + \varepsilon/4\},$$

$$S_\alpha^1(z) = \{w : g_\alpha(w,z) \leqq h(z) - \varepsilon/4\}.$$

Using the fact that there are only 2^ω compact subsets of $\mathbf{D}^E$, we can now take a subset σ of σ_2 with $|\sigma| = \tau$, such that K_α is the same subset K of $\mathbf{D}^E$ for all $\alpha \in \sigma$. Finally, let the real number r be chosen so that

$$h(z) - \varepsilon/4 \leqq r \leqq h(z)$$

for all z in some non-null subset L of K.

We now have, by Fubini's theorem, that when M is a finite subset of σ and $\varepsilon \in \mathbf{D}^M$,

$$\lambda_\tau\Big(\bigcap_{\alpha\in M} A_\alpha^{\varepsilon(\alpha)}\Big) \geqq \int_L \Big[\prod_{\alpha\in M} \lambda_{E(\alpha)\setminus E}(S_\alpha^{\varepsilon(\alpha)}(z))\Big] d\lambda_E(z).$$

Since the integrand is everywhere nonzero on the non-null set L, this integral is nonzero, as required.

2.4 THEOREM. *Let τ be a regular cardinal with the property that $\kappa^\omega < \tau$ whenever $\kappa < \tau$, and let T be a compact space. Then T carries a measure which is homogeneous of type τ if and only if there exists a continuous surjection from T onto* $[0,1]^\tau$.

PROOF. By virtue of 1.1 and 2.1, it will be enough to prove, under the assumption that T carries a homogeneous measure μ of type τ, that T contains an independent system $(A^0_\alpha, A^1_\alpha)_{\alpha\in\sigma}$ of closed sets, with $|\sigma| = \tau$.

Let $\Phi : L^1(\mu) \to L^1(\lambda_\tau)$ be an isometry (which we can assume to be positive, and isometric also from $L^\infty(\mu)$ to $L^\infty(\lambda_\tau)$). Denote by e_α the function $e_\alpha(z) = 1 - 2z_\alpha$ $(z \in \mathbf{D}^\tau)$. Then $\| e^{\cdot}_\alpha \|_\infty = 1$ and $\| e^{\cdot}_\alpha - e^{\cdot}_\beta \|_1 = 1$ $(\alpha \neq \beta)$.

For each α there is a function g_α in $C(T)$ with $\| g_\alpha \|_\infty = 1$ and

$$\| \Phi g^{\cdot}_\alpha - e^{\cdot}_\alpha \|_1 \leqq 1/4.$$

So the elements $f^{\cdot}_\alpha = \Phi g_\alpha$ of $L^\infty(\lambda_\tau)$ satisfy the hypotheses of 2.3.

If we obtain σ, r, δ as before and put

$$B^0_\alpha = \{t \in T : g_\alpha(t) \geqq r + \delta\},\ B^1_\alpha = \{t \in T : g_\alpha(t) \leqq r\} \qquad (\alpha \in \sigma),$$

the sets B^0_α, B^1_α are closed in T and

$$\mu\Big(\bigcap_{\alpha\in M} B^{\varepsilon(\alpha)}_\alpha\Big) = \lambda_\tau\Big(\bigcap_{\alpha\in M} A^{\varepsilon(\alpha)}_\alpha\Big) > 0,$$

whenever M is a finite subset of σ, and $\varepsilon \in D^M$.

2.5 REMARKS. Consider the following properties of a compact space T and an infinite cardinal τ:

(i) there is a continuous surjection from T onto $[0,1]^\tau$;

(ii) $C(T)$ has a subspace isometric to $l^1(\tau)$;

(iii) $C(T)$ has a subspace isomorphic to $l^1(\tau)$;

(iv) $C(T)^*$ has a subspace isomorphic to $L^1(\lambda_\tau)$;

(v) $C(T)^*$ has a subspace isometric to $L^1(\lambda_\tau)$;

(vi) T carries a homogeneous measure of type at least τ;

(vii) T carries a homogeneous measure of type exactly τ.

The following implications, and to the best of my knowledge no others, are known to hold without restrictions on the cardinal τ:

$$\text{(i)} \Leftrightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Leftrightarrow \text{(vi)} \Rightarrow \text{(vii)}.$$

(For the implication (iv) $\Rightarrow$ (vi) one uses the Maharam theorem, and an argument (due to Lindenstrauss) to be found on page 221 of [7].) All seven properties are equivalent if τ has the property of 1.3, since the proof of Theorem 2.4 does in fact show that (vi) implies (i).

2.6 THEOREM. *Let τ be a regular cardinal with the property that $\kappa^\omega < \tau$ whenever $\kappa < \tau$, and let X be a Banach space. Then X has a subspace isomorphic to $l^1(\tau)$ if and only if X^* has a subspace isomorphic to $L^1(\lambda_\tau)$.*

PROOF. The implication in one direction is 2.2. So let us suppose that $E : L^1(\lambda_\tau) \to X$ is an isomorphic embedding. If we write T for the compact space ball X^*, under the weak* topology, and I for the natural embedding $X \to C(T)$, we can, using lemma 2 of [4] again, "lift" E to an embedding $F : L^1(\lambda_\tau) \to C(T)^*$, with $I^*F = E$. So, by (iv) $\Leftrightarrow$ (vi) of 2.5, T carries a homogeneous measure μ of type at least τ.

By the Stone–Weierstrass theorem, the closed sublattice of $C(T)$ generated by IX is exactly

$$C_0(T) = \{f \in C(T) : f(0) = 0\}.$$

If $J_1 : C(T) \to L^1(\mu)$ is the natural mapping,

$$J_1 C_0(T) \quad \text{is dense in} \quad L^1(\mu).$$

Thus the sublattice generated by $J_1 IX$ is dense in $L^1(\mu)$, and so the density character of $J_1 IX$ (in $L^1(\mu)$) is at least τ. Using the fact that $\mathrm{cf}(\tau) = \tau > \omega$, we conclude that there is a family $(x_\alpha)_{\alpha \in \tau}$ in ball X such that

$$\| J_1 I x_\alpha - J_1 I x_\beta \| \geqq \varepsilon > 0 \qquad (\alpha \neq \beta).$$

Using Proposition 2.3, we know there exist a subset σ of τ with $|\sigma| = \tau$ and a real number r such that the sets

$$B^0_\alpha = \{\xi \in T : \langle \xi, x_\alpha \rangle \geqq r + \varepsilon/4\},$$

$$B^1_\alpha = \{\xi \in T : \langle \xi, x_\alpha \rangle \leqq r\}$$

form an independent family $(B^0_\alpha, B^1_\alpha)_{\alpha \in \sigma}$. By Proposition 1.2, the family $(x_\alpha)_{\alpha \in \sigma}$ is equivalent to the usual basis of $l^1(\sigma)$.

REMARK. As far as I am aware, it is another open question whether the conclusion of 2.6 is valid without the restrictions on the cardinal τ.

3. Two examples

Rosenthal showed in [8] that an infinite bounded subset of a Banach space contains either a weak Cauchy sequence or a sequence which is equivalent to the usual basis of l^1. It would be desirable to have a criterion of a similar kind which would enable us, under suitable conditions, to obtain embeddings of $l^1(\tau)$ spaces. The first example exhibited in this section will show that we should not hope for too much in this direction.

3.1 THEOREM. *There exist a Banach space X and an uncountable subset F of*

X such that F contains no weak Cauchy sequence, while X^* has no subspace isomorphic to $L^1(\lambda_{\omega_1})$ (so that X certainly does not have a subspace isomorphic to $l^1(\omega_1)$).

Hagler has recently shown this to be true, using an example of "James' tree" type [3]. The author hopes there may still be some interest in the following construction, which shows that X may be chosen to be $C(S)$ for a suitable compact and totally disconnected space S. The set F will consist of the indicator functions of a family $(U_\alpha)_{\alpha\in\omega_1}$ of open and closed subsets of S. We shall have to prove that there is no infinite sequence $(\alpha(n))$ such that $1_{U_{\alpha(n)}}$ converges pointwise on S, and that S does not carry a measure of uncountable type.

3.2 CONSTRUCTION. Let us fix an injection $\alpha \mapsto t_\alpha$ of the set ω_1 into $\mathbf{R}$. Let S be the subset of $\mathbf{D}^{\omega_1}$ consisting of all $x = (x_\alpha)$ such that for no $\alpha, \beta, \gamma < \omega_1$ do we have

$$\alpha < \beta < \gamma,$$

$$t_\alpha > t_\beta < t_\gamma,$$

$$x_\alpha = 0,\ x_\beta = 1,\ x_\gamma = 0.$$

Define $U_\alpha = \{x \in S : x_\alpha = 1\}$ (so that 1_{U_α} is the coordinate function $f_\alpha(x) = x_\alpha$).

3.3 LEMMA. *If $(\alpha(n))$ is a strictly increasing sequence of ordinals in ω_1 and the sequence $(t_{\alpha(n)})$ is monotonic in R, then the sequence of functions $(f_{\alpha(n)})$ is independent (equivalently, the sequence $(U_{\alpha(n)}, S\setminus U_{\alpha(n)})$ is independent).*

PROOF. Given a sequence $\varepsilon = (\varepsilon(n))$ in $\mathbf{D}^\omega$, we shall show that there exists $x \in S$ with

$$x_{\alpha(n)} = \varepsilon(n) \qquad (n \in \omega).$$

Assume first that $(t_{\alpha(n)})$ is increasing. We define $x_{\alpha(n)} = \varepsilon(n)$ $(n \in \omega)$;
$x_\alpha = 1$, if $\alpha < \alpha(0)$, or, for some n, $\alpha(n) < \alpha < \alpha(n+1)$ and $t_{\alpha(n)} < t_\alpha$;
$x_\alpha = 0$, if $a \geqq \sup_m \alpha(m)$ or, for some n, $\alpha(n) < \alpha < \alpha(n+1)$ and $t_{\alpha(n)} > t_\alpha$.
We find that x is in S since, in fact, $t_\alpha < t_\beta$ whenever $\alpha < \beta$, $x_\alpha = 0$ and $x_\beta = 1$.
If $(t_{\alpha(n)})$ is decreasing, we define
$x_{\alpha(n)} = \varepsilon(n)$;
$x_\alpha = 0$, if $\alpha < \alpha(0)$, or, for some n, $\alpha(n) < \alpha(n+1)$ and $t_\alpha < t_{\alpha(n+1)}$;
$x_\alpha = 1$, if $\alpha \geqq \sup_m \alpha(m)$, or, for $\alpha(n) < \alpha < \alpha(n+1)$ and $t_\alpha > t_{\alpha(n+1)}$.

3.4 PROPOSITION. *The subset $F = \{f_\alpha : \alpha < \omega_1\}$ of $C(S)$ contains no weak Cauchy sequence.*

PROOF. Given any infinite subset M of ω_1, we can find a strictly increasing sequence $(\alpha(n))$ in M such that $(t_{\alpha(n)})$ is monotonic. Thus every sequence in F has an independent subsequence, and F has no weak Cauchy sequence.

To show that there is no measure on S of uncountable type, we employ two further lemmas.

3.5 LEMMA. *Let Y be a compact metric space, and Z be a compact, totally ordered space. Then every atomless measure on $Y \times Z$ is of type ω.*

PROOF. Let μ be an atomless probability measure on $Y \times Z$ and let ν be the marginal probability on Z, $\nu(B) = \mu(Y \times B)$. We need only consider the case where ν is also atomless. For each rational q in $(0, 1)$, choose $z_q \in Z$ such that

$$\nu(\{z \in Z : z < z_q\}) = q.$$

If (B_n) is a countable base for the topology of Y, define, for each triple (n, q, r) with $n \in \omega$, $q, r \in \mathbf{Q} \cap (0,1)$ and $q < r$, the rectangle $R(n, q, r) = B_n \times \{z \in Z : z_q < z < z_r\}$. Then the set of indicator functions $1_{R(n,q,r)}$ is total in $L^1(\mu)$, and this space is thus separable.

Our last lemma enables us to decribe the subsets of our space S which are supports of measures. Recall that a space T has the *countable chain condition* (CCC) if every disjoint collection of nonempty open subsets of T is countable. The support of a measure necessarily has the CCC.

3.5 LEMMA. *Let T be a subset of S, and suppose that T has the CCC. Then T is homeomorphic to a subset of $Y \times Z$ for a suitable compact metric Y and compact, totally ordered Z.*

PROOF. I assert that there is an ordinal $\mathcal{T} < \omega_1$ such that, for $\alpha, \beta \geqq \mathcal{T}$ and $x \in T$,

$$x_\alpha = 0, \quad x_\beta = 1 \quad \text{imply} \quad t_\alpha < t_\beta.$$

Suppose the contrary; we construct open sets V_ξ $(\xi < \omega_1)$ as follows.

Take $\mathcal{T}(0) = 0$. If $\mathcal{T}(\xi) < \omega_1$ has been defined, choose $\alpha, \beta > \mathcal{T}(\xi)$ and $x \in T$ such that $x_\alpha = 0$, $x_\beta = 1$, and $t_\alpha > t_\beta$.

Put $V_\xi = \{y \in T : y_\alpha = 0, y_\beta = 1\}$ and let I_ξ be the interval (t_β, t_α) in R. To complete the inductive definition, we must choose $\mathcal{T}(\xi + 1)$ with $\alpha, \beta < \mathcal{T}(\xi + 1) < \omega_1$, and put $\mathcal{T}(\eta) = \sup\{\mathcal{T}(\xi) : \xi < \eta\}$ when η is a limit ordinal.

It is straightforward to check, from the above construction and the definition of S, that $V_\xi \cap V_\eta = \varnothing$ whenever $I_\xi \cap I_\eta \neq \varnothing$. Now there is an uncountable subset $\sigma \subset \omega_1$ and a rational q such that all the open intervals I_ξ $(\xi \in \sigma)$ contain

q. So $\{V_\xi : \xi \in \sigma\}$ is an uncountable disjoint collection of nonempty open subsets of T, contradicting the CCC.

We now take $\mathcal{T}$ to have the property stated above and define $Y = \mathbf{D}^{\mathcal{T}}$, $Z = \{z \in \mathbf{D}^{\omega_1 \setminus \mathcal{T}} : t_\alpha < t_\xi$ whenever $z_\alpha = 0,\ z_\beta = 1\}$. Then T is homeomorphic to a subset of $Y \times Z$, and Y is certainly metrizable (since $\mathcal{T}$ is countable). Moreover, Z can be identified with the set of all increasing $\{0, 1\}$-valued functions on the subset $\{t_\alpha : \mathcal{T} \leqq \alpha < \omega_1\}$ of $\mathbf{R}$, and it is, therefore, totally ordered.

The proof that the space S has the stated properties is now complete. The second example uses a construction explained to me by D. H. Fremlin. It is again concerned with a conjecture about the existence of $l^1(\omega_1)$ subspaces. Rosenthal has asked: "If there is a bounded sequence in X^* with no weakly* convergent subsequence, need X have a subspace which is (a) isomorphic to l^1; (b) isomorphic to $l^1(\omega_1)$?"

I shall show that the answer to question (b) is in the negative, and the counterexample will once more be a space of the type $C(S)$. As far as I am aware, question (a) is still unanswered, but it is rather easy to see that the answer is "yes" in the special case of $C(S)$-spaces.

3.7 THEOREM. *There is a compact space T which is not sequentially compact, but which carries no measure of uncountable type.*

PROOF. Let $\mathcal{R}$ be a family of subsets of $\mathbf{N}$, maximal with respect to the condition that for $R, R' \in \mathcal{R}$ at least one of the sets $R \setminus R'$, $R' \setminus R$, $R \cap R'$ is finite. (That is to say, of R and R', either one "almost contains" the other, or the two are "almost disjoint".) Evidently $\mathcal{R}$ contains all the finite subsets of $\mathbf{N}$. We take T to be the compactification of $\mathbf{N}$ determined by $\mathcal{R}$. We may view T either as the quotient space of βN by the appropriate equivalence relation, or as the closure of $\Phi\mathbf{N}$ in $\{0, 1\}^{\mathcal{R}}$ where $\Phi : \mathbf{N} \to \{0, 1\}^{\mathcal{R}}$ is the injection given by $\Phi(n) = (1_R(n))_{R \in \mathcal{R}}$. Either way, we can identify $\mathbf{N}$ with a dense open subset of T. For $R \in \mathcal{R}$, the closure $\bar{R}$ of R in T is open and closed in T, and the sets of this form, together with their complements, make up a subbase for the topology of T.

3.8 LEMMA. *No subsequence of $\mathbf{N}$ converges in T.*

PROOF. It is enough to show that for any infinite subset M of $\mathbf{N}$ there is a set $R \in \mathcal{R}$ with $M \cap R$ and $M \setminus R$ both infinite. We first choose a subset N of M such that N and $M \setminus N$ are both infinite. If $N \in \mathcal{R}$, we are finished; if not, the maximality of $\mathcal{R}$ tells us that there exists $R \in \mathcal{R}$ such that $N \cap R$ and $N \setminus R$ are both infinite.

We now suppose, if possible, that μ is a homogeneous measure on T, of uncountable type. Let S be the support of μ (so that S has the CCC) and let $\mathcal{S}$ be the set of all non-empty intersections $S \cap \tilde{R}$ with $R \in \mathcal{R}$. If $A, A' \in \mathcal{S}$, then either $A \subseteq A'$ or $A' \subseteq A$ or $A \cap A' = \emptyset$. Moreover, if $A, A' \in \mathcal{R}$ and $A \backslash A' \neq \emptyset$ then $\mu(A \backslash A') \neq 0$ (since $S = \operatorname{supp} \mu$, and $A \backslash A'$ is open in S).

3.9 LEMMA. *Let $\mathcal{D}$ be a maximal disjoint subset of $\mathcal{S}$. Then the set $\mathcal{D}^* = \{A \in \mathcal{S} : A \supseteq D$ for some $D \in \mathcal{D}\}$ is countable.*

PROOF. I assert first that a set $A \in \mathcal{D}^*$ is uniquely determined by $\mathcal{D}_A = \{D \in \mathcal{D} : D \subseteq A\}$. For suppose the sets $A, A' \in \mathcal{D}^*$ satisfy $\mathcal{D}_A = \mathcal{D}_{A'}$, $A' \subsetneq A$. Put $X = A \backslash A'$ and $\mathcal{T} = \{B \in \mathcal{S} : A' \subseteq B \subsetneq A\}$. It follows from maximality of the set $\mathcal{D}$ that if $C \in \mathcal{S}$ and $C \cap X$ is a proper nonempty subset of X then $C \in \mathcal{T}$. (For, if not, that is to say, if C does not contain A', we have $C \cap A' = \emptyset$ and $C \subseteq A$, which together imply that $C \cap D = \emptyset$ for all $D \in \mathcal{D}$.) Thus the sets $B \backslash A'$, $A \backslash B$ ($B \in \mathcal{T}$) form a subbase for the topology of X, and this topology is induced by a total order ($x \leqq y$ if $x \in B$ whenever $y \in B \in \mathcal{T}$). It follows that X must be null for any homogeneous measure of uncountable type. As we remarked earlier, $\mu(A \backslash A') = 0$ implies $A \backslash A' = \emptyset$.

It is now enough to prove that there are only countably many subsets of $\mathcal{D}$ of the form $\mathcal{D}_A$. This is easy when we note, firstly, that $\mathcal{D}$ is countable (by the CCC) and, secondly, that for a pair $\mathcal{D}_A \mathcal{D}_{A'}$, either $\mathcal{D}_A \subseteq \mathcal{D}_{A'}$, or $\mathcal{D}_{A'} \subseteq \mathcal{D}_A$ or $\mathcal{D}_A \cap \mathcal{D}_{A'} = \emptyset$.

We can now complete the proof of 3.7. For each n, let $\mathcal{D}_n$ be a maximal disjoint subset of $\mathcal{S}$, with the property that $\mu(D) \leqq 1/n$ for all $D \in \mathcal{D}_n$. (It is not hard to see that such $\mathcal{D}_n$ exist.) If $A \in \mathcal{S}$ and $A \not\in \mathcal{D}_n^*$ then $A \subseteq D$ for some $D \in \mathcal{D}_n$, so that $\mu(A) \leqq 1/n$. Since $\mu(A) > 0$ for all $A \in \mathcal{S}$, we have $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{D}_n^*$, and thus see that $\mathcal{S}$ is countable. It follows from this that S is metrizable and that S cannot, therefore, carry a measure of uncountable type.

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